Definitions for the usage of General Broken Lines for a TPC

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February 21, 2014

Abstract

The track re-fitting package General Broken Lines (GBL [1]) by C. Kleinwort needs specific information about the tracks and hits to be fitted. This is a summary for the TPC setup, as it is also used in an implementation for MarlinTPC [2].

1 Introduction

For the reconstruction of the trajectory of a charged particle in a tracking device several algorithms exist. In the absence of a magnetic field, charged particles ideally have a straight flight path. With a constant magnetic field and also in the absence of disturbances, their track is bent into a circle in the plane perpendicular to the magnetic field direction because of the Lorentz force. This is only true in theory, in a real detector the matter is more complicated. The most prominent deviation from this is caused by material along the flight path, causing multiple scattering or energy loss during the interaction.

In a gaseous detector like a TPC multiple scattering and energy loss play a small, usually negligible role. A larger effect arises from distortions of the magnetic and electric field, causing deviations from the ideal path. The distortions affect the path of the incident particle, e.g. in changes of curvature because of a changing magnetic field. In the TPC the largest effect is on the drift path of the primary electrons from their original creation towards the readout plane through the chamber.

The GBL takes an approach to deal with the multiple scattering in a straightforward way, by defining a natural coordinate system along the trajectory. At each scatter[1] and measurement plane this system is used to express the further propagation. To initialize

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[1]GBL uses thin scatterers, as any thick scatterer can be described by a pair of thin scatterers.

and interface the routines, the different systems and propagators need to be defined. In the description here no scattering material is included, as it is only used for the track evaluation within the gas volume. The description of scattering material is important when effects in the wall of the TPC play a role, e.g. at a test beam when the beam enters through one wall. Still, even without the need to include multiple scattering, an asset of the GBL is the direct interface to Millepede-II [3]. By providing additional information in terms of local or global derivatives it can be used to extract correction parameters, for example to the geometry which is usually used for alignment.

In order to use the GBL, several parts must be provided, which is the topic of this note. Most importantly these are the projections from the chosen track parameters in the local frame to the measurement system and the propagators from one point to the next. In the section 2 the basic equations are given, from which the projections (in section 3) and the propagators (in section 4) are derived.

2 Coordinate systems and Track parameters

In the context of ilcsoft and LCIO tracks are parametrized in perigee frame parameters with respect to the nominal detector origin[2] $\{\Omega, \tan\lambda, \Phi_0, d_0, z_0\}$. This is the same convention as the L3 experiment [4].

LCIO further defines the overall coordinate system (including the TPC) to be Cartesian. The track parameters are then the distances of the point of closest approach (p.c.a.) to the reference point d_0 in xy, and z_0 in z-direction. The track direction in xy at the p.c.a. is then Φ_0, the inclination in sz is $\tan\lambda$, where s is the arc length of the circle in the xy projection. The final parameter is then the signed curvature $\Omega = -\frac{1}{R}$, which is zero for straight line tracks. The sign is chosen such that a particle with negative charge will move anti-clockwise in a magnetic field along B_z, meaning the charge shares the sign with the direction of motion. In figures 1 and 2 a graphical representation of the parameters is shown.

The equations to describe all points on the trajectory in terms of these parameters are then:

$$\vec{x}(s) = \begin{pmatrix} d_0 \cos\Phi_0 + \frac{2}{\Omega} \sin\left(\frac{\Omega s}{2}\right) \cos\left(\Phi_0 - \frac{\Omega s}{2}\right) \\ d_0 \sin\Phi_0 + \frac{2}{\Omega} \sin\left(\frac{\Omega s}{2}\right) \sin\left(\Phi_0 - \frac{\Omega s}{2}\right) \\ z_0 + s \tan\lambda \end{pmatrix} \tag{1}$$

In the case of a straight line it is more straightforward to use s', the path length in 3D space. It is then simply $s = s' \cdot \cos\lambda$. The equations for the helix need to be modified to show the transition for $\Omega \to 0$. The term $\frac{2}{\Omega} \sin\left(\frac{\Omega s}{2}\right)$ becomes simply s in this limit. The equations then simplify to:

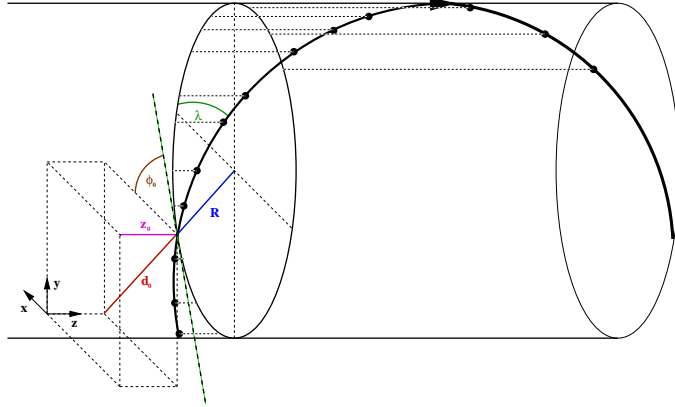[2]In other words: the reference point has the coordinates (0., 0., 0.).

Figure 1: A perspective sketch of the helix parameters is shown ([5]). d_0, ϕ_0 and Ω describe a circle in the xy-plane, where Ω is the inverse radius. The two remaining parameters describe a straight line in the sz plane.
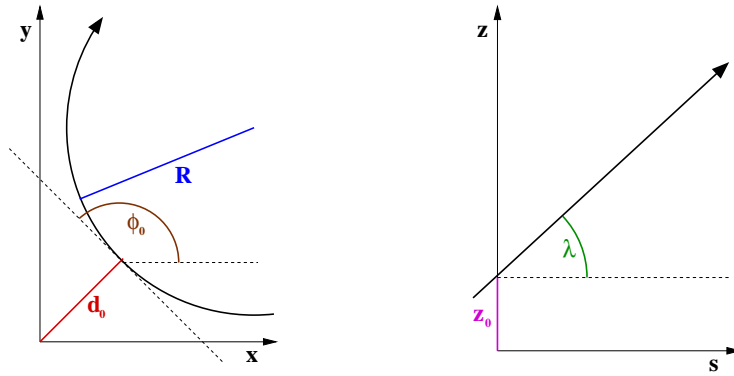

Figure 2: On the left the parametrization of a circle in the xy plane is shown with d_0, ϕ_0 and Ω ([5]). Ω is the curvature, which readily allows the limit of a straight line with $\Omega = 0$. On the right hand the two remaining parameters in the sz plane are shown. s is the path length in the xy plane.

$$\vec{x}(s') = \begin{pmatrix} d_0 \cos \Phi_0 + s' \cos \Phi_0 \cos \lambda \\ d_0 \sin \Phi_0 + s' \sin \Phi_0 \cos \lambda \\ z_0 + s' \sin \lambda \end{pmatrix} \tag{2}$$

To move from point to point along the trajectory, it is mathematically easier to use a different parametrization of the track at each point. Most practical for the description of multiple scattering, as in the GBL, is the choice of the curvilinear frame – a local coordinate system that follows the trajectory. For any point on the trajectory the parametric description can be derived from the above equations. Following the conventions of [6], the three directions $\{\vec{T}, \vec{U}, \vec{V}\}$ can be defined as following (the transverse plane to each point

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on the track is defined by the subset: $\vec{u} = \{\vec{U}, \vec{V}\}$)

$$\vec{T} = \frac{1}{\left|\frac{\mathrm{d}\vec{x}(s)}{\mathrm{d}s}\right|}\frac{\mathrm{d}\vec{x}(s)}{\mathrm{d}s}, \qquad \vec{U} = \hat{e}_z \times \vec{T}, \qquad \vec{V} = \frac{\vec{T} \times \vec{U}}{|\vec{T} \times \vec{U}|} \tag{3}$$

Please note, that in general this system depends in all components on the path length s – so more accurately written it is: $\{\vec{T}(s), \vec{U}(s), \vec{V}(s)\}$.

In the helix case, this leads to the following expressions:

$$\vec{T} = \begin{pmatrix} \cos(\Phi_0 - \Omega s)\cos\lambda \\ \sin(\Phi_0 - \Omega s)\cos\lambda \\ \sin\lambda \end{pmatrix} \tag{4}$$

$$\vec{U} = \begin{pmatrix} -\sin(\Phi_0 - \Omega s) \\ \cos(\Phi_0 - \Omega s) \\ 0 \end{pmatrix} \tag{5}$$

$$\vec{V} = \begin{pmatrix} -\cos(\Phi_0 - \Omega s) \\ -\sin(\Phi_0 - \Omega s) \\ \cos\lambda \end{pmatrix} \tag{6}$$

And again, the degeneration for the straight line (as can be seen here directly from $\Omega = 0$); one can note that directions are constant now, i.e. no longer depending on the path length:

$$\vec{T} = \begin{pmatrix} \cos\Phi_0\cos\lambda \\ \sin\Phi_0\cos\lambda \\ \sin\lambda \end{pmatrix} \tag{7}$$

$$\vec{U} = \begin{pmatrix} -\sin\Phi_0 \\ \cos\Phi_0 \\ 0 \end{pmatrix} \tag{8}$$

$$\vec{V} = \begin{pmatrix} -\cos\Phi_0\sin\lambda \\ -\sin\Phi_0\sin\lambda \\ \cos\lambda \end{pmatrix} \tag{9}$$

The transformation of the track parameters and their errors from point to point are then described in these parameters. Regarding the parametrization of the transport, there are different possibilities when using GBL. The one most useful for a TPC uses slopes and offsets with respect to the local curvilinear frame defined above:

$$[q/p, \vec{u}', \vec{u}]$$

Another commonly used parametrization are the curvilinear track parameters, which use the track angles instead of local slopes. This parametrization is used in the later description of the point to point transport in section 4:

$$[q/p, \lambda, \phi, x_\perp, y_\perp]$$

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3 Measurement system and transformations

Input data (e.g. space points) are given in the global (Cartesian) coordinate system in LCIO. The coordinates of measurements and their respective errors need to be transformed to the local measurement system, and most importantly to the actual measurement errors along the true direction of the measurement. TPC measurements are not directly placed on readout surfaces, but *virtual* surfaces – projections of the actual readouts in the axis of the drift (usually the z-axis). Depending on the type of readout, these surfaces are planes or cylinder surfaces. Here only cylinder surfaces are presented, the typical measurement topology in a TPC.

The radius of the cylinder is fixed due to the position of the readout structure, e.g. a row of pads with constant distance to the origin. The actual measurement directions are then directly along the other two unit vectors of a cylindrical coordinate system. The z-measurement follows the main TPC principle of the projection of a measured time along the drift path along the z-axis[3]. The other direction is then along φ, the azimuthal angle. Therefore the two measurement directions are

$$\hat{e}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \qquad \text{and} \qquad \hat{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{10}$$

A measurement has then the φ direction of its measured (or calculated) argument φ_p. The subscript p possibly denotes the measured space point *position* (or *pad*):

$$\vec{m} = \left\{ \vec{M}_1, \vec{M}_2 \right\} = \left\{ \begin{pmatrix} -\sin\varphi_p \\ \cos\varphi_p \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{11}$$

The projection matrix from measurement to the curvilinear coordinate system is then defined with

$$\vec{M}_i = (\vec{M}_i \cdot \vec{T})\vec{T} + (\vec{M}_i \cdot \vec{U})\vec{U} + (\vec{M}_i \cdot \vec{V})\vec{V} \tag{12}$$

For the projection only the parts transverse to the track direction are important, so with $\vec{u} = \{\vec{U}, \vec{V}\}$ the projection from measurement to local is

$$\vec{P}^m = \frac{\partial \vec{m}}{\partial \vec{u}} \qquad \text{with} \qquad \vec{P}^m_{ij} = \vec{M}_i \vec{U}_j \tag{13}$$

The other direction from local curvilinear to measurement is the inverted matrix

$$\vec{P}^u = \frac{\partial \vec{u}}{\partial \vec{m}} = \left(\vec{P}^m\right)^{-1} \tag{14}$$

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[3]The choice of this direction is pure convention; but there are so many sources of possible confusion, it's better to stick to this.

There are only two cases to be considered, the intersection of

1. a circle with a circle – the case with magnetic field. In the perpendicular projection the trajectory is a circle that intersects with the circle of the cylinder.

2. a circle with a straight line – the case without magnetic field. Here the circular diameter of the cylinder is intersected by the straight line trajectory.

So the case with magnetic field can be written as

$$
\begin{aligned}
\vec{P}^m &= \begin{pmatrix} -\sin\varphi_p & \cos\varphi_p & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin(\Phi_0 - \Omega s) & -\cos(\Phi_0 - \Omega s) \\ \cos(\Phi_0 - \Omega s) & -\sin(\Phi_0 - \Omega s) \\ 0 & \cos\lambda \end{pmatrix} \\
&= \begin{pmatrix} \cos(\varphi_p - \Phi_0 + \Omega s) & \sin(\varphi_p - \Phi_0 + \Omega s)\sin\lambda \\ 0 & \cos\lambda \end{pmatrix}
\end{aligned}
\tag{15}
$$

And for the straight line (again the special case of $\Omega = 0$):

$$
\begin{aligned}
\vec{P}^m &= \begin{pmatrix} -\sin\varphi_p & \cos\varphi_p & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin\Phi_0 & -\cos\Phi_0\sin\lambda \\ \cos\Phi_0 & -\sin\Phi_0\sin\lambda \\ 0 & \cos\lambda \end{pmatrix} \\
&= \begin{pmatrix} \cos(\varphi_p - \Phi_0) & \sin(\varphi_p - \Phi_0)\sin\lambda \\ 0 & \cos\lambda \end{pmatrix}
\end{aligned}
\tag{16}
$$

4 Transport equation from point to point

The last part needed by the GBL is the mathematical description of the way how to get from one point to another. This is done by provision of the point to point Jacobian, in local track parameters (chosen/described at the end of section 2). It expresses the changes of each parameter from one point to the next. All information about the fields enter in this, e.g. the magnetic field. There are different choices possible for the local track parameters. For a TPC, the most suited choice is $\vec{p}_i = \{q/p, \vec{u}', \vec{u}\} = (q/p, u_1', u_2', u_1, u_2)^T$:

- Signed curvature q/p ($\propto 1/R$)

- Direction, e.g. the slope $\vec{u}' = \partial\vec{u}/\partial w$ with w the path length[4] of the trajectory between adjacent points

- Offsets $\vec{u} = \{u_1, u_2\}$ in the local curvilinear coordinates

This means for the helix case an (almost) full 5×5 propagation matrix from point $i-1$ to i. Note that there are two directions, plus (+) and minus (-), from which the point can

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[4]In section 2 this is denominated by s', but for clearer notation here another symbol is used.

approached. In the GBL each point has a Jacobian attached that describes how to get there from a previous point:

$$J_i^+ = \frac{\partial \vec{p}_i}{\partial \vec{p}_{i-1}} \tag{17}$$

Since the order of points can simply be inverted, the choice of direction is arbitrary in a sense; it only needs to be consistent over a track.

The choice of a particular parametrization needs to be done for the exact calculation. In [6] the curvilinear track parametrization is chosen, which is most beneficial for the evaluation of the error propagation. While in principle it is possible to transform the formulae to the LCIO parametrization, it is easier to perform the transformation prior to the actual calculation. Please refer to the next section on transformations for this definition.

The full Jacobian for the helix case is slightly more involved, but the full Jacobian is needed. Since basically no scattering occurs in the TPC volume, any approximation would determine the track model. The following is quoted completely from [6]. $\vec{M}(s)$ is the vector to a point on the helix at path length s. $\vec{H} = \vec{B}/|\vec{B}|$ is the normalized magnetic field vector. To simplify notation, $\vec{N} = (\vec{H} \times \vec{T})/\alpha$, with $\alpha = |\vec{H} \times \vec{T}|$, $\gamma = \vec{H} \cdot \vec{T}$, $Q = -|\vec{B}|q/p$ and $\theta = Q \cdot s$.

Then the non zero terms for the transition from $i = 0$ to the next are:

$$\frac{\partial(q/p)}{\partial(q/p_0)} = 1 \tag{18}$$

$$\frac{\partial \lambda}{\partial(q/p_0)} = -\alpha Q \left(\frac{q}{p}\right)^{-1} \cdot (\mathbf{NV})[\mathbf{T} \cdot (\mathbf{M_0} - \mathbf{M})] \tag{19}$$

$$\frac{\partial \lambda}{\partial \lambda_0} = \cos\theta(\mathbf{V_0} \cdot \mathbf{V}) + \sin\theta((\mathbf{H} \times \mathbf{V_0}) \cdot \mathbf{V}) + (1 - \cos\theta) \cdot (\mathbf{H} \cdot \mathbf{V_0}) \cdot (\mathbf{H} \cdot \mathbf{V})$$
$$+ \quad \alpha(\mathbf{N} \cdot \mathbf{V})[-\sin\theta(\mathbf{V_0 T}) + \alpha(1 - \cos\theta)(\mathbf{V_0 N}) - (\theta - \sin\theta)(\mathbf{H} \cdot \mathbf{T})(\mathbf{H} \cdot \mathbf{V_0})] \tag{20}$$

$$\frac{\partial \lambda}{\partial \phi_0} = \cos\lambda_0 \left\{ \cos\theta(\mathbf{U_0} \cdot \mathbf{V}) + \sin\theta((\mathbf{H} \times \mathbf{U_0}) \cdot \mathbf{V}) + (1 - \cos\theta) \cdot (\mathbf{H} \cdot \mathbf{U_0}) \cdot (\mathbf{H} \cdot \mathbf{V}) \right.$$
$$+ \quad \alpha(\mathbf{N} \cdot \mathbf{V})[-\sin\theta(\mathbf{U_0 T}) + \alpha(1 - \cos\theta)(\mathbf{U_0 N}) - (\theta - \sin\theta)(\mathbf{H} \cdot \mathbf{T})(\mathbf{H} \cdot \mathbf{U_0})]\} \tag{21}$$

$$\frac{\partial \lambda}{\partial x_{\perp 0}} = -\alpha Q(\mathbf{N} \cdot \mathbf{V})(\mathbf{U_0} \cdot \mathbf{T}) \tag{22}$$

$$\frac{\partial \lambda}{\partial y_{\perp 0}} = -\alpha Q(\mathbf{N} \cdot \mathbf{V})(\mathbf{V_0} \cdot \mathbf{T}) \tag{23}$$

$$\frac{\partial \phi}{\partial(q/p_0)} = -\frac{\alpha Q}{\cos\lambda} \left(\frac{q}{p}\right)^{-1} \cdot (\mathbf{NU})[\mathbf{T} \cdot (\mathbf{M_0} - \mathbf{M})] \tag{24}$$

$$\frac{\partial \phi}{\partial \lambda_0} = \frac{1}{\cos\lambda} \left\{ \cos\theta(\mathbf{V_0} \cdot \mathbf{U}) + \sin\theta((\mathbf{H} \times \mathbf{V_0}) \cdot \mathbf{U}) + (1 - \cos\theta) \cdot (\mathbf{H} \cdot \mathbf{V_0}) \cdot (\mathbf{H} \cdot \mathbf{U}) \right.$$
$$+ \quad \alpha(\mathbf{NU})[-\sin\theta(\mathbf{V_0 T}) + \alpha(1 - \cos\theta)(\mathbf{V_0 N}) - (\theta - \sin\theta)(\mathbf{H} \cdot \mathbf{T})(\mathbf{H} \cdot \mathbf{V_0})]\} \tag{25}$$

$$\frac{\partial \phi}{\partial \phi_0} = \frac{\cos\lambda_0}{\cos\lambda} \left\{ \cos\theta(\mathbf{U_0} \cdot \mathbf{U}) + \sin\theta((\mathbf{H} \times \mathbf{U_0}) \cdot \mathbf{U}) + (1 - \cos\theta) \cdot (\mathbf{H} \cdot \mathbf{U_0}) \cdot (\mathbf{H} \cdot \mathbf{U}) \right.$$
$$+ \quad \alpha(\mathbf{NU})[-\sin\theta(\mathbf{U_0 T}) + \alpha(1 - \cos\theta)(\mathbf{U_0 N}) - (\theta - \sin\theta)(\mathbf{H} \cdot \mathbf{T})(\mathbf{H} \cdot \mathbf{U_0})]\} \tag{26}$$

$$\frac{\partial \phi}{\partial x_{\perp 0}} = -\frac{\alpha Q}{\cos\lambda}(\mathbf{N} \cdot \mathbf{U})(\mathbf{U_0} \cdot \mathbf{T}) \tag{27}$$

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$$\frac{\partial \phi}{\partial y_{\perp 0}} = -\frac{\alpha Q}{\cos \lambda}(\mathbf{N} \cdot \mathbf{U})(\mathbf{V_0} \cdot \mathbf{T}) \tag{28}$$

$$\frac{\partial x_{\perp}}{\partial (q/p_0)} = \left(\frac{q}{p}\right)^{-1}[\mathbf{U} \cdot (\mathbf{M_0} - \mathbf{M})] \tag{29}$$

$$\frac{\partial x_{\perp}}{\partial \lambda_0} = \frac{\sin \theta}{Q}(\mathbf{V_0} \cdot \mathbf{U}) + \frac{1 - \cos \theta}{Q}((\mathbf{H} \times \mathbf{V_0}) \cdot \mathbf{U}) + \frac{\theta - \sin \theta}{Q}(\mathbf{H} \cdot \mathbf{V_0})(\mathbf{H} \cdot \mathbf{U}) \tag{30}$$

$$\frac{\partial x_{\perp}}{\partial \phi_0} = \cos \lambda_0 \left\{ \frac{\sin \theta}{Q}(\mathbf{U_0} \cdot \mathbf{U}) + \frac{1 - \cos \theta}{Q}((\mathbf{H} \times \mathbf{U_0}) \cdot \mathbf{U}) + \frac{\theta - \sin \theta}{Q}(\mathbf{H} \cdot \mathbf{U_0})(\mathbf{H} \cdot \mathbf{U}) \right\} \tag{31}$$

$$\frac{\partial x_{\perp}}{\partial x_{\perp 0}} = (\mathbf{U_0} \cdot \mathbf{U}) \tag{32}$$

$$\frac{\partial x_{\perp}}{\partial y_{\perp 0}} = (\mathbf{V_0} \cdot \mathbf{U}) \tag{33}$$

$$\frac{\partial y_{\perp}}{\partial (q/p_0)} = \left(\frac{q}{p}\right)^{-1}[\mathbf{V} \cdot (\mathbf{M_0} - \mathbf{M})] \tag{34}$$

$$\frac{\partial y_{\perp}}{\partial \lambda_0} = \frac{\sin \theta}{Q}(\mathbf{V_0} \cdot \mathbf{V}) + \frac{1 - \cos \theta}{Q}((\mathbf{H} \times \mathbf{V_0}) \cdot \mathbf{V}) + \frac{\theta - \sin \theta}{Q}(\mathbf{H} \cdot \mathbf{V_0})(\mathbf{H} \cdot \mathbf{V}) \tag{35}$$

$$\frac{\partial y_{\perp}}{\partial \phi_0} = \cos \lambda_0 \left\{ \frac{\sin \theta}{Q}(\mathbf{U_0} \cdot \mathbf{V}) + \frac{1 - \cos \theta}{Q}((\mathbf{H} \times \mathbf{U_0}) \cdot \mathbf{V}) + \frac{\theta - \sin \theta}{Q}(\mathbf{H} \cdot \mathbf{U_0})(\mathbf{H} \cdot \mathbf{V}) \right\} \tag{36}$$

$$\frac{\partial y_{\perp}}{\partial x_{\perp 0}} = (\mathbf{U_0} \cdot \mathbf{V}) \tag{37}$$

$$\frac{\partial y_{\perp}}{\partial y_{\perp 0}} = (\mathbf{V_0} \cdot \mathbf{V}) \tag{38}$$

$$\tag{39}$$

The equations given here are the general case; where for a TPC in the given coordinate system only a magnetic field along the z-axis makes sense; such that $\mathbf{H} = \hat{e}_z$. Thus the equations can be simplified for programming, since the vector products can be written shorter.

For the straight line case the Jacobian is much simpler – the offset is simply the path length times the slope. The Jacobian for the same parameters as before $(q/p, \lambda, \phi, x_{\perp}, y_{\perp})$ then is only depending on the (three-dimensional) path length Δw:

$$J^+ = \frac{\partial \vec{p}_i}{\partial \vec{p}_{i-1}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \lambda \Delta w & 1 & 0 \\ 0 & 0 & 0 & \Delta w & 1 \end{pmatrix} \tag{40}$$

5   Transformations

The calculation of the Jacobians is performed in curvilinear track parameters. But the persistency format is the LCIO/L3 track parametrization. In [6] the transformation between a perigee frame and the curvilinear system is given. So only the transformation matrix between the therein given perigee frame and the LCIO parametrization is needed.

In [7] the original definition of $\{\kappa, \vartheta, \phi, \epsilon, z_p\}$ can be found, which leads to the conversion to $\{\Omega, \tan\lambda, \Phi_0, d_0, z_0\}$. The two transformations can be done in a single step (by multiplying the matrices); but here they are given separately for full reference.

The Jacobian of the transformation from the original perigee system to the curvilinear frame is given by ([6], A.28):

$$\frac{\partial(q/p, \lambda, \phi, x_\perp, y_\perp)}{\partial(\kappa, \vartheta, \phi, \epsilon, z_p)} = \begin{pmatrix} -\frac{\sin\vartheta}{B_z} & \frac{q}{p\tan\vartheta} & 0 & 0 & 0 \\ 0 & -1 & 0 & -\alpha Q(\mathbf{T}\cdot\mathbf{J})(\mathbf{V}\cdot\mathbf{N}) & -\alpha Q(\mathbf{T}\cdot\mathbf{K})(\mathbf{V}\cdot\mathbf{N}) \\ 0 & 0 & 1 & -\frac{\alpha Q(\mathbf{T}\cdot\mathbf{J})(\mathbf{U}\cdot\mathbf{N})}{\cos\lambda} & -\frac{\alpha Q(\mathbf{T}\cdot\mathbf{K})(\mathbf{U}\cdot\mathbf{N})}{\cos\lambda} \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & (\mathbf{V}\cdot\mathbf{K}) \end{pmatrix} \quad (41)$$

The other direction is the inverse of this expression (also given explicitly in [6], A.29). The change between the two different perigee systems is then straightforward. With $\Omega = -\kappa$, $\lambda = \pi/2 - \vartheta$, $\phi_0 = \phi$, $d_0 = -\epsilon$ and $z_0 = z_p$:

$$\frac{\partial(\Omega, \lambda, \Phi_0, d_0, z_0)}{\partial(\kappa, \vartheta, \phi, \epsilon, z_p)} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (42)$$

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