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Complete Two Loop Electroweak Contributions to the Muon Lifetime in the Standard Model

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Abstract

An independent result for the two loop fermionic contributions to the muon lifetime in the Standard Model is obtained and slight deviations in the prediction of the W boson mass are found with respect to [1]. Supplied with the bosonic contributions from [2–4], the shift, due to the complete electroweak contributions, varies from -2.4 MeV to -0.6 MeV. Additionally, a new test of the matching procedure defining the Fermi constant is presented, which uses fermion masses as infrared regulators.

1 Introduction

The muon lifetime is one of the key observables of today's particle physics. Not only is it measured very precisely, since the current experimental error is 18 ppm [5], but can be described to competing accuracy within the Standard Model, giving rise to a strong correlation between the masses of the heavy gauge bosons. As a low energy process, the decay is expected to be governed by an effective interaction involving only the electron, muon and their respective neutrinos. The dynamics of the system should be corrected mostly by QED, whereas the electroweak interactions determine solely the size of the coupling constant.

The history of the calculation of the electroweak corrections, in which we are interested here, is rather long. It started in the early eighties with the one loop contributions [6]. Subsequently, leading terms in the top quark $\mathcal{O}(\alpha^2 m_t^4)$ [7] and Higgs boson $\mathcal{O}(\alpha^2 M_H^2)$ [8] masses were derived at the two loop level. In the meantime, mixed electroweak and QCD corrections became available

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at order $\mathcal{O}(\alpha\alpha_s)$ [9] and $\mathcal{O}(\alpha\alpha_s^2)$ [10]. Recently, three loop leading terms in the top quark mass $\mathcal{O}(\alpha^3 m_t^6)$ and $\mathcal{O}(\alpha^2\alpha_s m_t^4)$ have also been calculated [11]. As far as the pure two loop electroweak corrections are concerned, after it turned out that the subleading terms in the top quark mass expansion are comparable with the leading ones [12], complete fermionic and the Higgs boson mass dependence of the bosonic contributions have been evaluated [1]. The complete bosonic part has been done in [2–4]. It is the purpose of the present paper to present the result of a new independent calculation of the fermionic contributions and, after inclusion of the bosonic corrections, also of the full electroweak corrections.

This work is organized as follows. In the next Section, we discuss the matching procedure and fermion masses as infrared regulators, which avoid ambiguities of the definition of gamma matrices in box diagrams in noninteger dimensions. Then, we present the results for the fermionic and full contributions and compare them with previous calculations by specifying the differences in the W boson mass prediction. Conclusions close the paper.

2 Matching

Due to a large number of very different mass scales, it is virtually impossible to evaluate the muon decay lifetime directly within the Standard Model, without recourse to some approximation method. An elegant and systematic approximation is provided by the approach based on *effective theories*. The idea is to agree on some cutoff scale, below which all degrees of freedom are treated exactly, whereas the heavier fields are "integrated out", which means that they generate effective interactions. It should not be surprising that one first discovers experimentally the effective theories, since the dependence on the heavier scales requires higher "resolution", *i.e.* higher energy. For precisely this reason, the effective theory governing muon decay, the Fermi Model, has been known much before the Standard Model. From this point of view, one should not consider that the Fermi Model is used in current calculations for *historical reasons*, but because it is the appropriate effective theory at this energy scale.

The approximation is constructed as follows. The lagrangian is made only from the light fields, which are the six leptons, the five quarks, the photon and the gluon. At leading order in the inverse heavy scale, for which we take the W boson mass M_W, a single effective operator is added, giving the lagrangian (in the so–called charge conserving form of the Fermi operator)

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{kin}}(\nu) + \mathcal{L}_{\text{QED}}(\alpha^0, m_l^0, m_q^0, l^0, q^0, A_\mu^0) + \mathcal{L}_{\text{QCD}}(\alpha_s^0, m_q^0, q^0, A_\mu^{a,0}) \quad (1)$$

$$+\frac{G_F}{\sqrt{2}}\ \overline{e^0}\gamma^\alpha(1-\gamma_5)\mu^0 \times \overline{\nu_\mu}\gamma_\alpha(1-\gamma_5)\nu_e,$$

where the superscript 0 denotes bare quantities. The theory is finite after mass and coupling (α and α_s) renormalization to all orders in α and α_s, and leading order in the Fermi constant $G_F \sim 1/M_W^2$, which is why this parameter is not renormalized.

The *matching procedure* in the present case consists in requiring that the amputated renormalized Green functions [1] of the effective theory be equal to the amputated renormalized Green functions of the Standard Model up to terms of order $\mathcal{O}(1/M_W^4)$ and given order in α and α_s

$$\mathcal{G}_{\rm SM} = \mathcal{G}_{\rm eff} + \mathcal{O}(1/M_W^4), \tag{2}$$

which makes the muon decay amplitude the same in both models up to the specified order. The Fermi constant is then given as an expansion in α and α_s

$$G_F = \sum_{i=0}^{\infty} G_F^{(i)} = \frac{\pi\alpha}{\sqrt{2}s_W^2M_W^2}(1+\Delta r), \tag{3}$$

with $G_F^{(0)} = \pi\alpha/(\sqrt{2}s_W^2M_W^2)$ being the Born level prediction. The quantity Δr is customarily used to parametrize the higher order contributions. At the one loop level, the matching equation is schematically depicted in Fig. 1. We introduced there the decoupling coefficients [13], $Z^*_{e,\mu}$, which are different from one in the $\overline{\rm MS}$ scheme for example, but can be neglected in the on-shell scheme. The renormalization constant of the Fermi operator Z_{O_F}, although trivial (*i.e.* equal to one), has been included for generality.

The matching equation, Eq. 2, can be solved in different ways. The apparently simplest is to put all light masses and external momenta to zero, and renormalize the wave functions in the on-shell scheme. The right hand side in Fig. 1 will then consist of only one term, proportional to $G_F^{(1)}$, if we use dimensional regularization, whereas the left will only have vacuum diagrams with heavy masses. Obviously, this situation will persist to all orders. The price to pay for this simplicity is the problem of infrared divergent box diagrams, where a product of gamma matrices occurs which does not have the form of the Fermi operator. In [2–4], this product has been defined through Fierz symmetry with respect to the last line in the string, which has been implemented, for practical reasons, by means of a suitable projection operator. Although sufficient at the two loop level, this symmetry will not suffice at the three loop level, see for

[1] This choice is somewhat arbitrary, since one might just as well use full Green functions, or Green functions which are one particle irreducible with respect to the light fields.

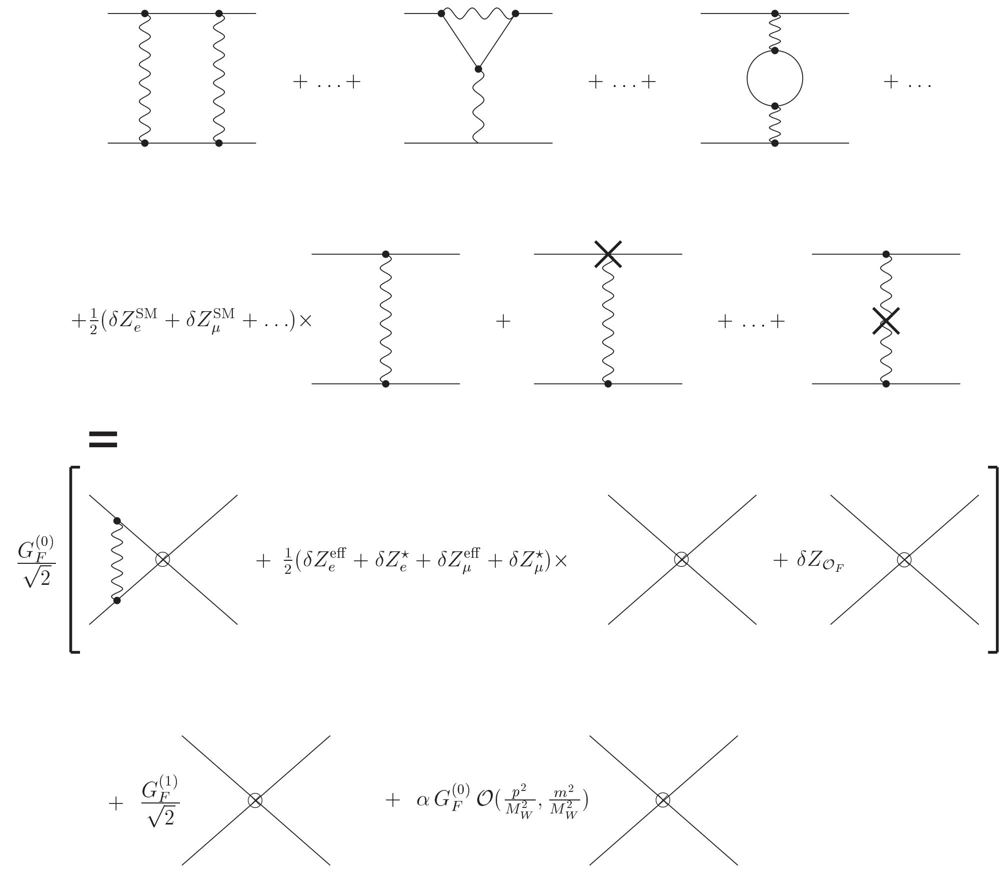

Fig. 1. Matching equation at the one loop level. The wavy lines on the left hand side represent the three gauge bosons, γ, W and Z, whereas on the right, only the photon. The rest of the notation is explained in the text.

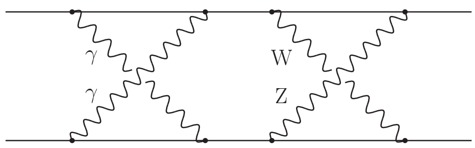

Fig. 2. A three loop diagram, which cannot be defined by Fierz symmetry with respect to the last line.

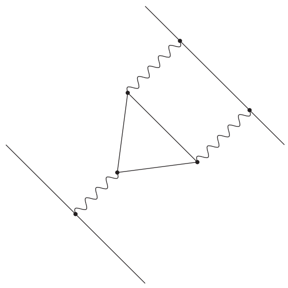

Fig. 3. A triangular femion loop requiring special treatment of the γ_5 matrix.

example Fig. 2, where due to crossings, there is no last line in this sense. It is also not trivial that this procedure is correct even at the two loop level. For many topologies, *e.g.* those that contain a self energy insertion on the gauge boson line, one can convince oneself that this is indeed the case, others like the nonplanar double boxes in the purely bosonic contributions are not that easy. A highly nontrivial test of the calculation would consist in performing the matching without generating spurious infrared divergences [2]. The box diagrams being ultraviolet finite can then be calculated in four dimensions, avoiding completely the problem of ambiguous gamma matrix definitions.

In this work we performed the matching by keeping a common mass for all the light fermions and evaluating the box diagrams in four dimensions. It was necessary to calculate both sides of the matching equation and reexpand them subsequently in this common mass. The external wave function renormalization constants were not taken in the on-shell scheme, because this would introduce the usual on-shell infrared divergence. On the contrary, they were evaluated at zero momentum, which, in practice, is equivalent to renormalization in the $\overline{\mathrm{MS}}$ scheme with nonvanishing decoupling coefficients. Moreover, it turned out that it is necessary to have a correct W boson wave function renormalization constant, since the box diagrams are not gauge invariant by themselves, and in the massive case this constant cancels only in combination with vertex diagrams. We used a photon mass regulator, but the $\overline{\mathrm{MS}}$ renormalization constant would have been just as good. In the end, complete agreement was found with the calculation performed with massless fermions and with the projector conserving Fierz symmetry with respect to the last line from [2–4].

3 Results

A detailed presentation of the methods used to evaluate the bosonic contributions to $\Delta r^{(\alpha^2)}$ can be found in [4]. The fermionic contributions introduce two additional problems. First, some of the two loop vertex diagrams contain closed triangular fermion loops as shown in Fig 3. The γ_5 matrix that occurs in the trace has to be correctly defined. We chose the naive dimensional regularization scheme [16], with an anticommuting γ_5 and the four dimensional value of the trace of four gamma matrices and γ_5

$$\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\gamma_5) = 4i\epsilon^{\alpha\beta\gamma\delta}. \tag{4}$$

This choice is justified by the fact that the nonvanishing contribution of the purely four dimensional ϵ tensors is finite. Moreover, it has been checked in [1] that the use of the consistent definition of 't Hooft and Veltman [17] gives the same result after correction of the Green functions by suitable finite counterterms restoring the Slavnov–Taylor identities. Second, the inclusion of fermions results in unstable gauge bosons, which makes a proper definition of their

[2] Note that one could also introduce evanescent operators [14] and perform the calculation with vanishing fermion masses and without projection, as for example in [15]. This would be a second independent test.

masses necessary if gauge invariance of G_F is to be maintained [18]. We use the pole mass scheme, where the inverse propagator matrix

$$(s - M_i^2)\delta_{ij} - \Pi_{T\,ij}(s), \quad i,j = W,\gamma,Z, \tag{5}$$

is singular in the complex s plane at points which can be parametrized as

$$s_{\mathrm{P}} = M_{\mathrm{P}}^2 - iM_{\mathrm{P}}\Gamma_{\mathrm{P}}, \tag{6}$$

where M_{P} is the mass and Γ_{P} is the width of the boson. This generates a fixed width Breit–Wigner behavior of the total cross section

$$\sigma(s) \sim \frac{1}{(s - M_{\mathrm{P}}^2)^2 + M_{\mathrm{P}}^2\Gamma_{\mathrm{P}}^2}, \tag{7}$$

as opposed to the running width parametrization actually used by the experimental collaborations for the masses and widths of the W and Z bosons [19]

$$\sigma(s) \sim \frac{1}{(s - M_{\mathrm{exp}}^2)^2 + s^2\Gamma_{\mathrm{exp}}^2/M_{\mathrm{exp}}^2}. \tag{8}$$

We translate back and forth between the two definition with the help of the following relations

$$M_{\mathrm{P}} = M_{\mathrm{exp}}\left(1 + \frac{\Gamma_{\mathrm{exp}}^2}{M_{\mathrm{exp}}^2}\right)^{-1/2}, \quad \Gamma_{\mathrm{P}} = \Gamma_{\mathrm{exp}}\left(1 + \frac{\Gamma_{\mathrm{exp}}^2}{M_{\mathrm{exp}}^2}\right)^{-1/2}. \tag{9}$$

As in [1], we take Γ_Z as experimentally measured, whereas we assume Γ_W to be given by the one loop QCD corrected value

$$\Gamma_W = \frac{3G_F M_W^3}{2\sqrt{2}\pi}\left(1 + \frac{2\alpha_s(M_W)}{3\pi}\right). \tag{10}$$

The complete result for Δr at order α^2 and the partial contributions are given in Fig. 4. The top quark mass and the running of the fine structure constant are taken from Tab. 1, whereas the masses of the gauge bosons are translated from the experimental values given there to the pole mass scheme values with the help of Eq. 9, which in this case gives $M_W = 80.424$ GeV and $M_Z = 91.1535$ GeV.

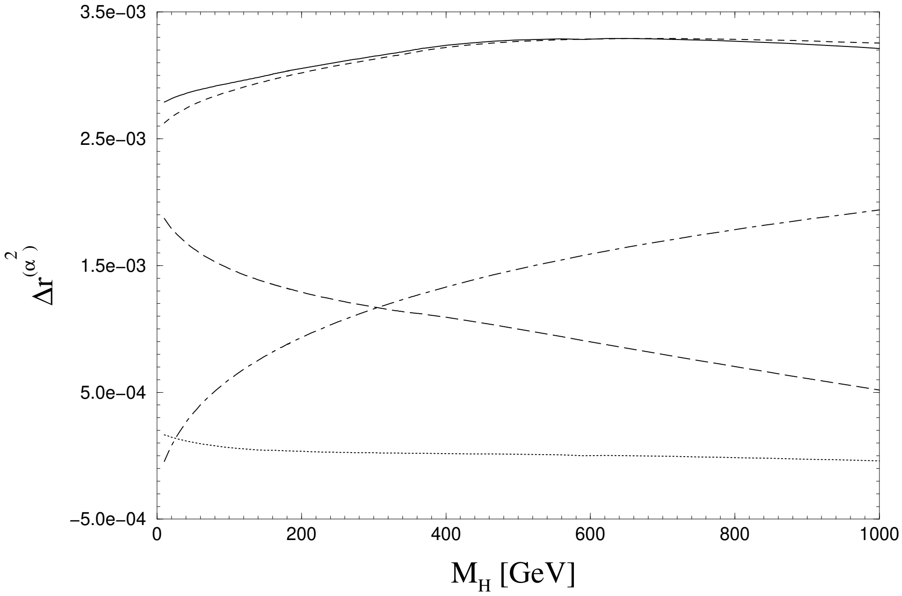

Fig. 4. Complete two loop electroweak contributions to Δr (solid line) together with partial corrections: bosonic (dotted line), fermionic (dashed line), light fermionic without b quark, but with running of the fine sturcture constant (dash–dotted line) and top–bottom (long dashed line).

input parameter	value	source
M_W	80.451(33) GeV	[5]
M_Z	91.1876 GeV	[5]
m_t	174.3(51) GeV	[5]
m_b	4.7 GeV	[1]
G_μ	1.16637×10^{-5} GeV^{-2}	[20]
α^{-1}	137.03599976	[5]
$\Delta\alpha$	0.059228(209)	[21]
$\alpha_s(M_Z)$	0.119	[5]
Γ_Z	2.4952 GeV	[5]

Table 1
Input parameters with experimental errors, where necessary for the present work. The value of m_b is the same as in [1] for comparison purposes.

In order to compare our result for the fermionic contributions with [1], we evaluate the W boson mass from the formula

$$M_W = M_Z \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\pi\alpha}{\sqrt{2} G_F M_Z^2}(1 + \Delta r)}}, \tag{11}$$

	$\Delta r^{(\alpha)} + \Delta r^{(\alpha^2)}_{\rm ferm}$	$+\,\Delta r^{(\alpha\alpha_s)} + \Delta r^{(\alpha\alpha_s^2)}$		$+\Delta r^{(2)}_{\rm bos}$	
M_H [GeV]	M_W [1] [GeV]	M_W [GeV]	ΔM_W [MeV]	M_W [GeV]	ΔM_W [MeV]
100	80.3771	80.3758	-1.3	80.3747	-2.4
200	80.3338	80.3326	-1.2	80.3321	-1.7
600	80.2521	80.2509	-1.2	80.2508	-1.3
1000	80.2135	80.2122	-1.3	80.2129	-0.6

Table 2
Comparison of the M_W prediction (third and fifth column) against [1] (second column). ΔM_W is, in both cases, the shift with respect to the fitting formula.

with

$$\Delta r = \Delta r^{(\alpha)} + \Delta r^{(\alpha\alpha_s)} + \Delta r^{(\alpha\alpha_s^2)} + \Delta^{(\alpha^2)}_{\rm ferm}. \tag{12}$$

We keep a finite b quark mass in $\Delta r^{(\alpha)}$ and $\Delta r^{(\alpha\alpha_s)}$ and take the result for $\Delta r^{(\alpha\alpha_s^2)}$ from [22]. Note also that we do not resum the running of the fine structure constant, *i.e.* $\Delta r^{(\alpha)}$ contains the term $+\Delta\alpha$ and $\Delta r^{(\alpha^2)}_{\rm ferm}$ includes $+\Delta\alpha^2$. The result is summarized in Tab. 2 for different Higgs boson masses from the range from 100 GeV to 1 TeV. We observe a discrepancy of around -1.3 MeV with respect to [1], which comes solely from the differing fermionic contributions [3].

Inclusion of the bosonic corrections generates an additional variable shift already given in [2,3]. As a result, our complete contributions induce a change of the M_W prediction by -2.4 MeV for a Higgs boson mass as low as 100 GeV (see Tab. 2). Since the bosonic part becomes negative for a heavier Higgs boson, this shift reaches -0.6 MeV for $M_H = 1$ TeV.

In Tab. 3, we include also the recent partial results at three loop order [11], *i.e.* we use

$$\Delta r = \Delta r^{(\alpha)} + \Delta r^{(\alpha\alpha_s)} + \Delta r^{(\alpha\alpha_s^2)} + \Delta^{(\alpha^2)} - \frac{c_W^2}{s_W^2}\left(\Delta\rho_t^{(\alpha^3)} + \Delta\rho_t^{(\alpha^2\alpha_s)}\right). \tag{13}$$

Together with errors coming from the top quark mass and the running of the fine structure constant but without a theoretical error estimate, the M_W

[3] The authors of [1] traced a problem in their calculation and after corrections agree with our results both for the fermionic and for the Higgs boson mass dependence of the bosonic two loop contributions. We checked that all of the remaining corrections are the same to required numerical accuracy.

		$-c_W^2/s_W^2 \Delta\rho_t^{(\alpha^3)}$		$-c_W^2/s_W^2 \Delta\rho_t^{(\alpha^2\alpha_s)}$	
M_H [GeV]	M_W [GeV]	M_W [GeV]	ΔM_W [MeV]	M_W [GeV]	ΔM_W [MeV]
100	80.3747	80.375	0.3	80.3771	2.4
200	80.3321	80.3322	0.1	80.3358	3.7
600	80.2508	80.2510	0.2	80.2579	7.1
1000	80.2129	80.2146	1.7	80.2231	10.2

Table 3
Additional shift of M_W with respect to the complete prediction from Tab. 2 due to inclusion of partial results at order α^3 and $\alpha^2\alpha_s$ from [11].

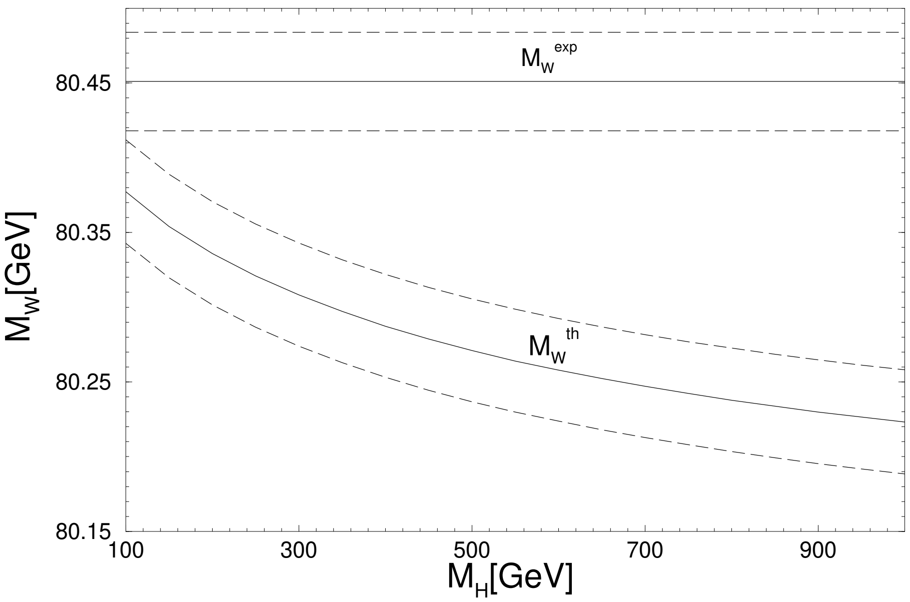

Fig. 5. The theoretical prediction for the W boson mass, $M_W^{\rm th}$, with error from the uncertainty of the top quark mass and the running of the fine structure constant, against the current experimental value, $M_W^{\rm exp}$.

prediction is shown against the current experimental result in Fig. 5.

4 Conclusions

We have presented a new result for the complete electroweak contributions to the lifetime of the muon, which induces a shift in the W boson mass prediction as large as -2.4 MeV for a light Higgs boson, of which -1.3 MeV come from a discrepancy with the previous calculation of the fermionic contributions [1] and the rest from the bosonic part. The authors of [1] corrected their evaluation [4] and are now in full agreement with this work. Together with recent results at the three loop level [11], this calls for an updated fitting formula. Such a formula will be given in a subsequent publication [24].

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