LC-TH-2004-032
hep-ph/0407224
HD-THEP-04-28

Optimal observables and phase-space ambiguities

O. Nachtmann[1] and F. Nagel[2]

Institut für Theoretische Physik, Philosophenweg 16, D-69120 Heidelberg, Germany

Abstract

Optimal observables are known to lead to minimal statistical errors on parameters for a given normalised event distribution of a physics reaction. Thereby all statistical correlations are taken into account. Therefore, on the one hand they are a useful tool to extract values on a set of parameters from measured data. On the other hand one can calculate the minimal constraints on these parameters achievable by any data-analysis method for the specific reaction. In case the final states can be reconstructed without ambiguities optimal observables have a particularly simple form. We give explicit formulae for the optimal observables for generic reactions in case of ambiguities in the reconstruction of the final state and for general parameterisation of the final-state phase space.

[1]email: O.Nachtmann@thphys.uni-heidelberg.de
[2]email: F.Nagel@thphys.uni-heidelberg.de

1 Introduction

At a future e^+e^- linear collider (LC) like TESLA [1] or CLIC [2], apart from the possible discovery of new particles, electroweak precision measurements of various observables will be an important task. In this way effects due to new physics at a scale far beyond the produced c.m. energy may be detected. The e^+e^-, the $\gamma\gamma$, the $e^-\gamma$, the e^-e^- and the Giga-Z mode can reveal complementary aspects. For instance, consider the three-gauge-boson vertices γWW and ZWW, which are highly restricted in the Standard Model (SM), see for instance [3]. In a form-factor approach with the most general Lorentz-invariant parameterisation these vertices are described by 28 real parameters if one allows also for imaginary parts [4]. In many extensions of the SM one obtains deviations of these triple-gauge-boson couplings (TGCs) from their SM values, for an overview of the literature see e.g. Sect. 1 of [5]. At present the experimental constraints on these couplings from LEP are rather weak and many couplings have yet to be measured [6, 7, 8]. At a future LC all TGCs can be measured with much higher precision, see [9, 10, 11] and references therein. Both longitudinal and transverse beam polarisation are considered to be feasible [12] and will be advantageous in many cases, see for example [13]. For most TGCs longitudinal polarisation is found to be the right choice [5], whereas one coupling can only be determined with transverse polarisation [14]. In a gauge invariant effective-Lagrangian approach for the gauge-boson sector new-physics effects can be parameterised by ten anomalous couplings [15, 16]. In such a framework, after spontaneous symmetry breaking also gauge-boson-fermion interactions and gauge-boson masses are modified by the anomalous couplings. Therefore constraints can be derived from observables measured at LEP1, SLC and LEP2. Also in this case the couplings can be determined with much higher precision at a future LC [16].

In all cases it is important to know how sensitive the event distribution of a certain reaction is to a given set of parameters, independent of the method that will be used in the future to analyse the data. Moreover statistical correlations should be taken into account, unless they are very small, in order to provide realistic results. Frequently one is dealing with a situation where the normalised event distribution is—at least approximately—a linear function of the parameters to be measured. The method of optimal observables [17, 9] is most directly applied for the case of a strictly linear function. In most parts of this paper we assume that the event distributions are to good approximation linear functions of the parameters to be measured. By means of optimal observables one can then compute the maximum constraints on this set of parameters for a given reaction and given event number while taking into account all statistical correlations. Apart from being a useful tool for theorists to estimate the sensitivity of a reaction to a set of parameters optimal observables have been used in experimental analyses to extract the parameters, for instance TGCs [6] or CP violating parameters in $e^+e^- \to \tau^+\tau^-$ [18]. But also methods to handle non-linear functions of the parameters to be measured have been devised [10] and successfully applied in experimental analyses, see e.g. [19, 8].

In case of many parameters with correlated errors, the sensitivity to different directions in parameter space is often not easy to survey. Here optimal observables have the advantage that discrete symmetry properties of the differential cross section can be exploited in order to eliminate correlations between couplings. For example the TGCs are classified into four symmetry groups, and couplings from different symmetry groups can be measured without correlations when optimal observables are used [9]. However within each symmetry group couplings are in general correlated. In [10] it is shown that the optimal observables are unique up to linear transformations.

Here we construct explicitly the optimal observables for generic ambiguities in the reconstruction of the final state. Such ambiguities can be either continuous, that is the number of measured variables is smaller than the number of variables required to specify the final state, or discrete, that is a reconstructed event can originate from two or more final states.

This work is organised as follows: In Sect. 2 we resume the properties of optimal observables as given in [9]. In Sect. 3 the optimal observables for generic ambiguities in the reconstruction of the final state are discussed. In Sect. 4 we recall that the method can be applied iteratively to determine parameters on which the differential cross section depends non-linearly. We present our conclusions in Sect. 5.

2 Optimal observables

In this section we give a resumé of the definition and properties of optimal observables. As it is convenient to have an illustrative example in mind we shall discuss the problem to measure anomalous contributions to the differential cross sections for the reactions $e^+e^- \to WW$ and $\gamma\gamma \to WW$. But we emphasise that our considerations are neither restricted to anomalous couplings nor to particular reactions. The optimal-observables method can be applied to any reaction where the differential cross section depends on a certain number of small parameters, which we generically denote by h_i and which are to be estimated. Furthermore, the method can be generalised to the case where the parameters to extract are not necessarily small [10]. Consider now the reaction $e^+e^- \to WW$ or $\gamma\gamma \to WW$ and let us assume that we describe it in the framework of the SM with the addition of small real anomalous constants h_i.

In an experiment one measures the differential cross section

$$S(\phi) \equiv \mathrm{d}\sigma/\mathrm{d}\phi, \tag{1}$$

where ϕ denotes the set of all measured phase-space variables. For instance, the fully differential cross section of the process $e^+e^- \to WW$ without transverse beam polarisation depends on five angles, which are in this case specified by ϕ, see [5]. In the same way the spin-averaged fully differential cross section of $\gamma\gamma \to WW$ with fixed photon energies depends on five angles, see [20]. In case of $e^+e^- \to WW$ as treated within a form-factor approach in [5] the anomalous parameters h_i are the 28 anomalous TGCs that parameterise deviations at the γWW and ZWW vertices from the SM. In case of $\gamma\gamma \to WW$ as treated within an effective-Lagrangian approach in [20] the h_i are the ten coefficients of certain dimension-six operators. Here non-zero anomalous couplings do not only lead to anomalous three- and four-gauge-boson couplings but also to deviations of the gauge-boson-fermion couplings and of the gauge-boson masses from their SM values. We distinguish between the information from the total cross section $\sigma = \int \mathrm{d}\sigma$ and from the normalised distribution S/σ of the events. Here we only investigate how well anomalous couplings can be extracted from the latter. It is possible to obtain constraints on these parameters also from the measurement of σ, see Sect. 3.1 of [10]. Those considerations remain unaffected in the presence of ambiguities in the reconstruction of the final state.

Expanding S in the anomalous couplings one can write

$$S(\phi) = S_0(\phi) + \sum_i S_{1i}(\phi)\, h_i \; + \; O(h^2), \tag{2}$$

where $S_0(\phi)$ is the cross section in the SM and the $S_{1i}(\phi)$ give the first-order modifications due to the anomalous couplings. We assume $S_0(\phi)$ and $S_{1i}(\phi)$ to be calculated from theory. Note that the variables ϕ need not specify the final state completely. In the analyses [5, 14] of $e^+e^- \to WW$ with one W decaying leptonically and the other one into two hadronic jets it was assumed that the jet charges cannot be identified, which results in a two-fold ambiguity. In such cases $S(\phi)$ is not the fully differential cross section—in our example it is the sum over two final states. In a case with ambiguities of this or more involved kind it is often not straightforward to calculate $S_0(\phi)$ and $S_{1i}(\phi)$. But these quantities must be known explicitly in order to construct the optimal observables. This problem is our main concern in this paper and will be addressed in the following section.

We now give a short resumé of the optimal-observables method. One way to extract the parameters h_i from the measured distribution (2) is to look for a suitable set of observables $\mathcal{O}_i(\phi)$ whose expectation values

$$E[\mathcal{O}_i] = \frac{1}{\sigma} \int \mathrm{d}\phi\, S(\phi)\, \mathcal{O}_i(\phi) \tag{3}$$

are sensitive to the dependence of $S(\phi)$ on the couplings h_i. To first order in the anomalous couplings we have

$$E[\mathcal{O}_i] = E_0[\mathcal{O}_i] + \sum_j c_{ij} h_j \; + \; O(h^2), \tag{4}$$

with

$$E_0[\mathcal{O}_i] \;=\; \frac{1}{\sigma_0} \int \mathrm{d}\phi\, S_0(\phi)\, \mathcal{O}_i(\phi)\;, \tag{5}$$

$$
\begin{aligned}
c_{ij} &= \frac{1}{\sigma_0}\int \mathrm{d}\phi\, \mathcal{O}_i(\phi)\, S_{1j}(\phi) - \frac{\sigma_{1j}}{\sigma_0^2}\int \mathrm{d}\phi\, S_0(\phi)\, \mathcal{O}_i(\phi)\;, && (6)\\
\sigma_0 &= \int \mathrm{d}\phi\, S_0(\phi)\;, && (7)\\
\sigma_{1j} &= \int \mathrm{d}\phi\, S_{1j}(\phi)\;. && (8)
\end{aligned}
$$

Here $E_0[\mathcal{O}_i]$ is the expectation value for zero anomalous couplings, and c_{ij} gives the sensitivity of $E[\mathcal{O}_i]$ to h_j. Solving (4) for the set of the h_j we get estimators for the anomalous couplings, whose covariance matrix is given by

$$
V(h) = \frac{1}{N}\, c^{-1} V(\mathcal{O})\, (c^{-1})^{\mathrm{T}}, \tag{9}
$$

where we use matrix notation. Here N is the number of events, and

$$
V(\mathcal{O})_{ij} = \frac{1}{\sigma_0}\int \mathrm{d}\phi\, S_0(\phi)\, \mathcal{O}_i(\phi)\, \mathcal{O}_j(\phi) - E_0[\mathcal{O}_i]\, E_0[\mathcal{O}_j] \quad + \quad O(h) \tag{10}
$$

is the covariance matrix of the observables, which we have expanded around its value in the SM. As observables we choose

$$
\mathcal{O}_i(\phi) = \frac{S_{1i}(\phi)}{S_0(\phi)}. \tag{11}
$$

From (6) and (10) one obtains for this specific choice of observables

$$
V(\mathcal{O}) = c \;+\; O(h), \tag{12}
$$

and therefore

$$
V(h) = \frac{1}{N}\, c^{-1} \;+\; O(h). \tag{13}
$$

From (12) we see that for the observables (11) c is a symmetric matrix because $V(\mathcal{O})$ is symmetric. The observables (11) are "optimal" in the sense that for $h_i \to 0$ the errors (13) on the couplings are as small as they can be for a given probability distribution, see [9]. For details on this so-called Rao-Cramér-Fréchet bound, see for example [21]. Apart from being useful for actual experimental analyses, the observables (11) thus provide insight into the sensitivity that is at best attainable by *any* method, given a certain process and specified experimental conditions. In case of one parameter this type of observable was first proposed in [17], the generalisation to several parameters was made in [9]. Moreover, it has been shown that optimal observables are unique up to a linear reparameterisation [10]. We further note that phase-space cuts, as well as detector efficiency and acceptance have no influence on the observables being "optimal" in the above sense, since their effects drop out in the ratio (11). This is not the case for detector-resolution effects, but the observables (11) are still close to optimal if such effects do not significantly distort the differential distributions S_{1i} and S_0 (or tend to cancel in their ratio). To the extent that they are taken into account in the data analysis, none of these experimental effects will bias the estimators.

When a set of events is calculated with a Monte-Carlo generator and the events are then reconstructed like in an experimental analysis the optimal observables (11) are obtained directly in the measured variables ϕ. However for theoretical studies, i.e. to estimate the sensitivity of a certain reaction to anomalous couplings, and also for experimental analyses often the analytical expressions of the optimal observables (11) are required.

Frequently there are ambiguities in phase space, that is to one value of the measured kinematic variables ϕ there correspond two or more distinct final states (discrete ambiguities) or a bunch of final states (continuous ambiguities). The calculation of $S_0(\phi)$, $S_{1i}(\phi)$ in (2) and in particular of the observables $\mathcal{O}_i(\phi)$ in (11) has then to be done with some care as will be shown below.

3 Phase-space ambiguities

In principle there are plenty of possibilities to parameterise a final state in a reaction uniquely, for instance the usage of angles or Cartesian coordinates, different choices of reference frames etc. In an experiment one may either be able to specify a final state of an event uniquely or only with certain ambiguities. Here we discuss in detail the case of discrete ambiguities, that is for each event one only knows that it belongs to a group of two, three or more final states. We also mention how to handle continuous ambiguities, i.e. the case in which the number of measured variables is smaller than the number of variables required to specify the final state. An example of a discrete ambiguity is the two-fold one of the semileptonic final states in $e^+e^- \to WW$ or in $\gamma\gamma \to WW$ with fixed c.m. energy of the two-photon system. Here one usually assumes that the two hadronic jets cannot be associated unambiguously to the quark and antiquark. Another more involved one occurs in the reaction $\gamma\gamma \to WW$ when the photons each obey a Compton spectrum. Here, in addition to the ambiguity above, another two-fold one arises from the reconstruction of the neutrino momentum. This case is considered in [20]. The optimal observables (11) are now to be calculated in the presence of such ambiguities.

We start from a particular set of phase-space variables χ that specify the final state uniquely. The differential cross section in terms of these variables we denote by

$$T(\chi) \equiv \mathrm{d}\sigma/\mathrm{d}\chi. \tag{14}$$

The cross section for another choice of variables ϕ, that may lead to the above ambiguities, is then given by

$$S(\phi) = \int \mathrm{d}\chi \, \delta(F(\chi) - \phi) \, T(\chi). \tag{15}$$

The function F expressing the relation of ϕ to χ may take the same value for different values of χ, that is for a given ϕ the equation

$$F(\chi) = \phi \tag{16}$$

may have several solutions $\chi_k \equiv \chi_k(\phi)$ with $k = 1, 2, \ldots$. In general, the number of solutions to (16) may vary with ϕ. If ϕ are the coordinates that can be measured of an event χ, the set of final states χ_k consists of χ itself as well as all final states that cannot be distinguished from χ by a measurement of ϕ. Notice that in (15) we have assumed that the number of components of the vectors ϕ and χ are the same. In general there can be also continuous ambiguities, that is the fully differential cross section is specified by a larger number of variables than those that can actually be measured. However this does not lead to any further complicacies in the context of optimal observables. In fact, if the fully differential cross section is $\tilde{T}(\chi, \xi)$ where the final-state variables ξ cannot be measured, the generalisation of (15) is

$$S(\phi) = \int \mathrm{d}\chi \mathrm{d}\xi \; \delta(F(\chi) - \phi) \, \tilde{T}(\chi, \xi). \tag{17}$$

If we define

$$T(\chi) \equiv \int \mathrm{d}\xi \; \tilde{T}(\chi, \xi) \tag{18}$$

we again obtain (15). We can thus apply all formulae in the remainder of this section also in case of discrete *plus* continuous ambiguities. We remark that our analysis also works if one or more of the phase-space variables take discrete values as is the case for instance for spin indices. For these variables integrals have to be substituted by sums and δ-distributions by Kronecker symbols.

An integration and summation over part of the phase-space variables as in (17) is, of course, performed when one considers inclusive cross sections. Thus our discussion covers also this case. Clearly, then the normalisation integral $\int \mathrm{d}\phi \; S(\phi)$ gives the cross section times the corresponding multiplicity and in our formulae σ has to be read in this way.

Coming back to (15) we have

$$S(\phi) = \sum_k |J_k|^{-1} \, T(\chi_k(\phi)) \tag{19}$$

where

$$J_k \equiv \det \frac{\partial F}{\partial \chi}(\chi_k(\phi)) \tag{20}$$

is the Jacobian determinant taken at point χ_k. If F is invertible, there is only one term in the sum for all ϕ and (19) simplifies to

$$S(\phi) = \left| \frac{\partial F}{\partial \chi} \left(F^{-1}(\phi) \right) \right|^{-1} T \left(F^{-1}(\phi) \right). \tag{21}$$

We expand the differential cross section:

$$T(\chi) = T_0(\chi) + \sum_i T_{1i}(\chi)\, h_i \quad + \quad O(h^2). \tag{22}$$

It follows

$$S(\phi) = S_0(\phi) + \sum_i S_{1i}(\phi)\, h_i \quad + \quad O(h^2), \tag{23}$$

where

$$S_0(\phi) \;\; = \;\; \sum_k |J_k|^{-1}\, T_0(\chi_k(\phi)), \tag{24}$$

$$S_{1i}(\phi) \;\; = \;\; \sum_k |J_k|^{-1}\, T_{1i}(\chi_k(\phi)). \tag{25}$$

Note, again, that the number of terms in the sums (24) and (25) can vary with ϕ. If ϕ—but not necessarily χ—are coordinates that can be measured we have to define the optimal observables from the expansion of $S(\phi)$ in (23):

$$\mathcal{O}_i(\phi) = \frac{S_{1i}(\phi)}{S_0(\phi)}. \tag{26}$$

In the specific case where F is invertible going from χ to ϕ is a mere change of coordinates and we obtain the same optimal observables using either set of variables:

$$\mathcal{O}_i(\phi) = \left.\frac{T_{1i}\,(\chi)}{T_0\,(\chi)}\right|_{\chi = F^{-1}(\phi)}. \tag{27}$$

If there are ambiguities in the reconstruction but if we have the same Jacobian $J \equiv J_k$ for all k (which may nevertheless depend on ϕ), J cancels in the numerator and denominator of the observables (26):

$$\mathcal{O}_i(\phi) = \frac{\sum_k T_{1i}\,(\chi_k(\phi))}{\sum_k T_0\,(\chi_k(\phi))}. \tag{28}$$

This is the case e.g. for the reaction $e^+e^- \to WW$, where one W boson decays into a quark-antiquark pair and the other one into a lepton pair, if the charges of the two jets in the final state cannot be identified, see [9]. If this is not the case we must use the general expressions (24) to (26).

The covariance matrix of the observables (26) is now

$$V(\mathcal{O})_{ij} = \frac{1}{\sigma_0} \int_A \mathrm{d}\phi\; S_0(\phi)\, \mathcal{O}_i(\phi)\, \mathcal{O}_j(\phi) - \frac{\sigma_{1i}\sigma_{1j}}{\sigma_0^2} \quad + \quad O(h), \tag{29}$$

where

$$\sigma_0 \;\; \equiv \;\; \int_A \mathrm{d}\phi\; S_0(\phi) = \int_B \mathrm{d}\chi\; T_0(\chi), \tag{30}$$

$$\sigma_{1i} \;\; \equiv \;\; \int_A \mathrm{d}\phi\; S_{1i}(\phi) = \int_B \mathrm{d}\chi\; T_{1i}(\chi), \tag{31}$$

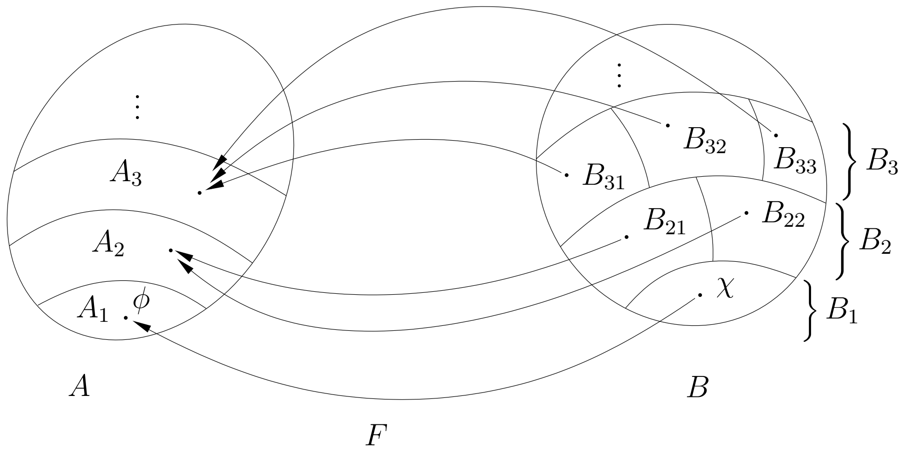

Figure 1: Definition of integration areas.

and the full kinematically allowed integration regions in the coordinates ϕ and χ are denoted by A and B, respectively. The integrals σ_0 and σ_{1i} can be performed in either coordinates. Using χ no knowledge about ambiguities in the reconstruction is necessary. The first term in the expression of $V(\mathcal{O})$ needs special care. We divide the integration region A into parts A_n with $n = 1, 2, \ldots$, such that for $\phi \in A_n$ there are n solutions χ_k to (16). The domains of B corresponding to the A_n we denote by B_n, see Fig. 1. We subdivide B_n into n appropriate regions B_{nk} with $k = 1, 2, \ldots, n$, such that $\chi_k \in B_{nk}$. This subdivision is certainly not unique. We have

$$\begin{aligned} H_{ij} &\equiv \int_A \mathrm{d}\phi\, S_0(\phi)\, \mathcal{O}_i(\phi)\, \mathcal{O}_j(\phi) \\ &= \int_A \mathrm{d}\phi\, \frac{S_{1i}(\phi) S_{1j}(\phi)}{S_0(\phi)} \\ &= \sum_{n\geq 1} \int_{A_n} \mathrm{d}\phi\, \frac{S_{1i}(\phi) S_{1j}(\phi)}{S_0(\phi)}. \end{aligned} \tag{32}$$

Expressed in terms of integrals over χ we get

$$H_{ij} = \int_{B_1} \mathrm{d}\chi\, \frac{T_{1i}(\chi)\, T_{1j}(\chi)}{T_0(\chi)} + \sum_{n\geq 2} \int_{B_{npn}} \mathrm{d}\chi\, |J(\chi)| \frac{S_{1i}\,(F(\chi))\, S_{1j}\,(F(\chi))}{S_0\,(F(\chi))}, \tag{33}$$

and also

$$H_{ij} = \int_{B_1} \mathrm{d}\chi\, \frac{T_{1i}(\chi)\, T_{1j}(\chi)}{T_0(\chi)} + \sum_{n\geq 2} \frac{1}{n} \int_{B_n} \mathrm{d}\chi\, |J(\chi)| \frac{S_{1i}\,(F(\chi))\, S_{1j}\,(F(\chi))}{S_0\,(F(\chi))}, \tag{34}$$

where we have from (24), (25) for $\chi \in B_n$

$$S_0\,(F(\chi)) = \sum_k^n \Big| J\Big(\chi_k(F(\chi))\Big)\Big|^{-1} T_0\Big(\chi_k(F(\chi))\Big), \tag{35}$$

$$
\begin{aligned}
S_{1i}\left(F(\chi)\right) &= \sum_{k}^{n} \left| J\Big(\chi_k(F(\chi))\Big) \right|^{-1} T_{1i}\Big(\chi_k(F(\chi))\Big), && (36) \\
J(\chi) &= \det \frac{\partial F}{\partial \chi}(\chi). && (37)
\end{aligned}
$$

One of the values $\chi_k(F(\chi))$ in (35), (36) is, of course, identical to χ. In (33) one can choose for each n any natural number p_n with $1 \le p_n \le n$. These choices correspond to different parameterisations of the integration regions A_n but leave the integrals unchanged. Therefore one may sum over all possible choices and divide by n, which leads to (34). The quantities H_{ij} may be calculated either in the form (33) or (34). Notice that the form (34) has the advantage that one only has to know where in the integration region for χ there are how many solutions to (16), but one does not have to specify B_{n1}, B_{n2}, etc. In certain cases the integrals for $n \ge 2$ in (33) or (34) may be simplified. For example let A'_n with $n \ge 2$ be the part of A_n where the Jacobians $J(\chi_k(\phi))$ are the same for all k. The Jacobian in this region may nevertheless depend on ϕ. The region of A_n where they are not the same for all k we call A''_n. The corresponding regions of B_n are denoted by B'_n and B''_n. We write the integrals in (34) as

$$
\int_{B_n} \mathrm{d}\chi = \int_{B'_n} \mathrm{d}\chi + \int_{B''_n} \mathrm{d}\chi. \qquad (38)
$$

Then, in the integrals over B'_n the Jacobian cancels and we obtain the following expression for the integral in the covariance matrix (29):

$$
\begin{aligned}
H_{ij} &= \int_{B_1} \mathrm{d}\chi \, \frac{T_{1i}(\chi)\, T_{1j}(\chi)}{T_0(\chi)} && (39) \\
&+ \sum_{n \ge 2} \frac{1}{n} \int_{B'_n} \mathrm{d}\chi \, \frac{\sum_k^n T_{1i}\Big(\chi_k(F(\chi))\Big) \sum_l^n T_{1j}\Big(\chi_l(F(\chi))\Big)}{\sum_m^n T_0\Big(\chi_m(F(\chi))\Big)} \\
&+ \sum_{n \ge 2} \frac{1}{n} \int_{B''_n} \mathrm{d}\chi \, |J(\chi)| \frac{S_{1i}\left(F(\chi)\right) S_{1j}\left(F(\chi)\right)}{S_0\left(F(\chi)\right)}
\end{aligned}
$$

with $S_0(F(\chi))$ and $S_{1i}(F(\chi))$ as in (35) and (36), respectively.

4 Iterative analysis in the non-linear case

In this section we recall briefly that the use of optimal observables is not restricted to a phase-space distribution depending only linearly on parameters h_i which are to be estimated. In other words, the higher-order terms in h_i in (2) can be handled. This has been discussed extensively in [10]. We recall here only one practical procedure one can follow.

Suppose we have a theoretical expression for the differential cross section (2) which can be expanded in the h_i:

$$S(\phi) = S_0(\phi) + \sum_i S_{1i}(\phi)\, h_i + \sum_{ij} S_{2ij}(\phi)\, h_i h_j \ + \ \ldots . \tag{40}$$

With given data an estimate of the h_i has to be made. The procedure proposed in [10] is then as follows. In the first step the terms of second and higher order are neglected and one follows the procedure of estimating the h_i by the optimal observables (11). Suppose that this gives as estimates for the parameters the values $\tilde{h}_i$. In the second step one sets

$$h_i = \tilde{h}_i + h'_i \tag{41}$$

and substitutes this for $S(\phi)$ in (40). This gives

$$S(\phi) = \tilde{S}_0(\phi) + \sum_i \tilde{S}_{1i}(\phi)\, h'_i + \sum_{ij} \tilde{S}_{2ij}(\phi)\, h'_i h'_j \ + \ \ldots , \tag{42}$$

where

$$\tilde{S}_0(\phi) \ = \ S_0(\phi) + \sum_i S_{1i}(\phi)\, \tilde{h}_i + \sum_{ij} S_{2ij}(\phi)\, \tilde{h}_i \tilde{h}_j \ + \ \ldots , \tag{43}$$

$$\tilde{S}_{1i}(\phi) \ = \ S_{1i}(\phi) + 2 \sum_j S_{2ij}(\phi)\, \tilde{h}_j \ + \ \ldots , \tag{44}$$

and so on.

Now one applies the optimal-observables method to estimate the h'_i, neglecting terms of second and higher order in the h'_i in (42). The new optimal observables are

$$\tilde{\mathcal{O}}_i(\phi) = \frac{\tilde{S}_{1i}(\phi)}{\tilde{S}_0(\phi)}. \tag{45}$$

Let $\tilde{h}'_i$ be the estimates for these parameters obtained in this way. The improved estimate for the original parameters is then $\tilde{h}_i + \tilde{h}'_i$.

This procedure can be iterated. It was tested in [19] in a Monte-Carlo study for the analysis of TGCs at LEP2. Parameters h_i—not necessarily small—were assumed and Monte-Carlo data generated according to the corresponding distribution (40) which for this case contained linear and quadratic terms in the h_i. The non-linear optimal-observables analysis was then performed as mentioned above. It turned out that after a few, typically three, iterations the input values for the h_i were obtained back within their correct statistical errors.

Clearly, the phase-space ambiguities are to be treated in exactly the way discussed in Sect. 3 also for the case of a non-linear analysis.

5 Conclusions

For electroweak precision measurements at a future LC it will be important to check the validity of the SM (or perhaps another theory) with the highest possible precision. To this end optimal observables are a convenient means because parameters can be determined with minimal errors as allowed by a theorem from mathematical statistics without neglecting correlations between any of them. Such observables have for instance been applied to the reaction $e^+e^- \to WW$ in [9, 10] with a form-factor approach to the γWW and ZWW vertices. Effects of beam polarisation to the same process were analysed in [5, 14]. However in the mentioned studies the final state was assumed to be known either exactly or up to a two-fold ambiguity of a very simple type. It was possible to add the respective terms of the differential cross section in order to construct the optimal observables. In this paper we have discussed the calculation of optimal observables for the case of an arbitrary reaction with generic ambiguities in the reconstruction of the final states. This is the case e.g. in the reaction $\gamma\gamma \to WW$ where the photons are not monochromatic but have a Compton-energy spectrum [20]. In the most general case the expressions for the optimal observables and the covariance matrix are somewhat complicated, basically because they contain the Jacobian determinant of the parameter transformation. However simplifications occur in various special cases, for example if the Jacobian determinant is a constant in phase space and therefore cancels in certain ratios and integrals. Using our results one is able to study the best statistically achievable sensitivity to a set of parameters in a reaction given a certain event number, if there are ambiguities in the reconstruction of the final state. Such studies give important information on the capabilities of future machines and on how to choose the experimental settings like polarisations in an optimal way for physics studies. Apart from that optimal observables are an ideal tool for the analysis of experimental data. In either case one often needs explicit analytical expressions of the optimal observables. To this end we have collected here the necessary formulae for the case when the final state cannot be fully reconstructed from the measured variables.

Acknowledgements

The authors are grateful to M. Diehl for reading a draft version of the manuscript and for good suggestions and to M. Pospischil for useful discussions. This work was supported by the German Bundesministerium für Bildung und Forschung, BMBF project no. 05HT4VHA/0, and the Deutsche Forschungsgemeinschaft through the Graduiertenkolleg "Physikalische Systeme mit vielen Freiheitsgraden".

References

[1] "TESLA Technical Design Report Part I: Executive Summary," eds. F. Richard, J. R. Schneider, D. Trines and A. Wagner, DESY, Hamburg, 2001 [hep-ph/0106314];
"TESLA Technical Design Report Part III: Physics at an e^+e^- Linear Collider," eds. R.-D. Heuer, D. Miller, F. Richard, P. M. Zerwas, DESY, Hamburg, 2001 [hep-ph/0106315];
"TESLA Technical Design Report, Part VI, Chapter 1: Photon collider at TESLA," B. Badelek *et al.*, DESY, Hamburg, 2001, [hep-ex/0108012].

[2] J. R. Ellis, E. Keil and G. Rolandi, "Options for Future Colliders at CERN," CERN-EP-98-03;
J. P. Delahaye *et al.*, "CLIC—a Two-Beam Multi-TeV e^+e^- Linear Collider," in: *Proc. of the 20th Intl. Linac Conference LINAC 2000*, ed. Alexander W. Chao, eConf C000821, MO201 (2000) [physics/0008064];
"Physics at the CLIC Multi-TeV Linear Collider: Report of the CLIC Physics Working Group," eds. M. Battaglia *et al.*, CERN, Geneva, 2004, CERN-2004-005;
H. Burkhardt and V. Telnov, "CLIC 3-TeV Photon Collider Option," CERN-SL-2002-013-AP.

[3] O. Nachtmann: "Elementary Particle Physics: Concepts and Phenomena". *Berlin, Germany: Springer* (1990).

[4] K. Hagiwara, R. D. Peccei, D. Zeppenfeld and K. Hikasa, Nucl. Phys. B **282**, 253 (1987).

[5] M. Diehl, O. Nachtmann and F. Nagel, Eur. Phys. J. C **27**, 375 (2003) [hep-ph/0209229].

[6] P. Abreu *et al.* [DELPHI Collaboration], Phys. Lett. B **459**, 382 (1999);
M. Acciarri *et al.* [L3 Collaboration], Phys. Lett. B **467**, 171 (1999) [hep-ex/9910008];
G. Abbiendi *et al.* [OPAL Collaboration], Eur. Phys. J. C **19**, 1 (2001) [hep-ex/0009022].

[7] M. Acciarri *et al.* [L3 Collaboration], Phys. Lett. B **487**, 229 (2000) [hep-ex/0007005];
G. Abbiendi *et al.* [OPAL Collaboration], Eur. Phys. J. C **19**, 229 (2001) [hep-ex/0009021].

[8] A. Heister *et al.* [ALEPH Collaboration], Eur. Phys. J. C **21**, 423 (2001) [hep-ex/0104034].

[9] M. Diehl and O. Nachtmann, Z. Phys. C **62**, 397 (1994).

[10] M. Diehl and O. Nachtmann, Eur. Phys. J. C **1**, 177 (1998) [hep-ph/9702208].

[11] W. Menges, "A study of charged current triple gauge couplings at TESLA," LC-PHSM-2001-022.

[12] G. Moortgat-Pick, in: *Proc. of the APS/DPF/DPB Summer Study on the Future of Particle Physics (Snowmass 2001)* eds. R. Davidson and C. Quigg, hep-ph/0202082;
K. Mönig, LC-PHSM-2000-059, in *2nd ECFA/DESY Study 1998-2001, p. 1346*;
R. W. Assmann and F. Zimmermann, "Polarization Issues at CLIC," in: *Proc. of the APS/DPF/DPB Summer Study on the Future of Particle Physics (Snowmass 2001)*, eds. R. Davidson and C. Quigg, CERN-SL-2001-064-AP;
G. Alexander et al., SLAC-Proposal-E-166, Stanford, 2003.

[13] G. Moortgat-Pick and H. M. Steiner, Eur. Phys. J. direct C **3**, 6 (2001) [hep-ph/0106155];
J. Erler *et al.*, in: *Proc. of the APS/DPF/DPB Summer Study on the Future of Particle Physics (Snowmass 2001)*, ed. N. Graf, eConf **C010630**, E3004 (2001) [hep-ph/0112070];
G. Moortgat-Pick, AIP Conf. Proc. **675**, 206 (2003) [hep-ph/0303234];
see also the POWER working group homepage:
http://www.ippp.dur.ac.uk/˜gudrid/power/, working group report in preparation.

[14] M. Diehl, O. Nachtmann and F. Nagel, Eur. Phys. J. C **32**, 17 (2003) [hep-ph/0306247].

[15] W. Buchmüller and D. Wyler, Nucl. Phys. B **268**, 621 (1986).

[16] O. Nachtmann, F. Nagel and M. Pospischil, "Anomalous gauge-boson couplings and the Higgs-boson mass," hep-ph/0404006.

[17] D. Atwood and A. Soni, Phys. Rev. D **45**, 2405 (1992);
M. Davier, L. Duflot, F. Le Diberder and A. Rougé, Phys. Lett. B **306**, 411 (1993).

[18] K. Ackerstaff *et al.* [OPAL Collaboration], Z. Phys. C **74**, 403 (1997);
N. Wermes, Nucl. Phys. Proc. Suppl. **55C**, 313 (1997).

[19] S. Dhamotharan: "Untersuchung des Drei-Eichbosonen-Vertex in W-Paarerzeugung bei LEP2", Doctoral thesis, HD-IHEP 99-04, Univ. Heidelberg, 1999.

[20] O. Nachtmann, M. Pospischil and F. Nagel, manuscripts in preparation.

[21] H. Cramér: "Mathematical Methods of Statistics". *Princeton, USA: Univ. Pr.* (1958).